

Ergodic Theory - Week 5

Course Instructor: Florian K. Richter
Teaching assistant: Konstantinos Tsinas

1 Birkhoff's pointwise ergodic theorem

P1. Let $u = [a_1, a_2, \dots] \in (0, 1)$, and let $\frac{p_n}{q_n}$ be its n -th convergent. Show that for any $n \in \mathbb{N}$

$$\left| u - \frac{p_n}{q_n} \right| > \frac{1}{q_n q_{n+2}}.$$

P2. A real number $u = [a_1, a_2, \dots] \in (0, 1)$ is called *badly approximable* if there exists some $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \geq 1$. Show that $u \in (0, 1)$ is badly approximable if and only if there exists some $\epsilon > 0$ such that

$$\left| u - \frac{p}{q} \right| \geq \frac{\epsilon}{q^2}$$

for all rational numbers $\frac{p}{q}$.

P3. (Strong Law of Large Numbers) Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, and consider $(X_n)_{n \in \mathbb{N}}$ a sequence of independent and identically distributed (i.i.d.) Lebesgue integrable random variables, with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\mu(\cdot) = \mathbb{P}(X_0 \in \cdot)$ the probability distribution of X_i . Consider the space $X = \mathbb{R}^{\mathbb{N}}$ equipped with the product sigma-algebra $\mathcal{B}$, the product measure $\nu = \bigotimes_{n \in \mathbb{N}} \mu$, and the left shift $(x_i)_{i \in \mathbb{N}} \in X \rightarrow \sigma((x_i)_{i \in \mathbb{N}}) = (x_{i+1})_{i \in \mathbb{N}} \in X$.

- (a) Justify that $(X, \mathcal{B}, \nu, \sigma)$ is a measure-preserving system.
- (b) Show that $(X, \mathcal{B}, \nu, \sigma)$ is mixing (see Exercise sheet 2 for the definition). Conclude, in particular, that it is ergodic.
- (c) Conclude that

$$\frac{X_0 + \dots + X_{N-1}}{N} \xrightarrow[N \rightarrow \infty]{} \mathbb{E}(X_0), \quad \mathbb{P} - \text{a.e.}$$